Find the value of each expression using the given information.

13. csc θ and tan θ; cos $\theta = \frac{3}{5}$, sin $\theta < 0$

SOLUTION:

Use the Pythagorean Identity that involves cos θ to find sin θ.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \sin^2\theta + \left(\frac{3}{5}\right)^2 &= 1 \\ \sin^2\theta + \frac{9}{25} &= 1 \\ \sin^2\theta &= \frac{16}{25} \\ \sin\theta &= \pm\frac{4}{5} \end{aligned}$$

Since sin $\theta < 0$, $\sin\theta = -\frac{4}{5}$. Use the reciprocal identity csc $\theta = \frac{1}{\sin\theta}$ to find csc θ.

$$\begin{aligned} \csc\theta &= \frac{1}{\sin\theta} \\ &= \frac{1}{-\frac{4}{5}} \\ &= -\frac{5}{4} \end{aligned}$$

Use the quotient identity tan $\theta = \frac{\sin\theta}{\cos\theta}$ to find tan θ.

$$\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} \\ &= \frac{-\frac{4}{5}}{\frac{3}{5}} \\ &= -\frac{4}{3} \end{aligned}$$

14. cot θ and cos θ; tan $\theta = \frac{2}{7}$, csc $\theta > 0$

SOLUTION:

Use the reciprocal function cot $\theta = \frac{1}{\tan\theta}$ to find cot θ.

$$\begin{aligned}\cot\theta &= \frac{1}{\tan\theta}\\ &= \frac{1}{\frac{2}{7}}\\ &= \frac{7}{2}\end{aligned}$$

Use the Pythagorean identity that involves tan θ to find sec θ.

$$\begin{aligned}\tan^2\theta + 1 &= \sec^2\theta\\ \left(\frac{2}{7}\right)^2 + 1 &= \sec^2\theta\\ \frac{53}{49} &= \sec^2\theta\\ \pm\frac{\sqrt{53}}{7} &= \sec\theta\end{aligned}$$

Since csc θ is positive, sin θ is positive. Since tan $\theta = \frac{\sin\theta}{\cos\theta}$ is positive and sin θ is positive, cos θ has to be positive. Since cos θ is positive, sec θ is positive. So, sec $\theta = \frac{\sqrt{53}}{7}$. Use the reciprocal identity cos $\theta = \frac{1}{\sec\theta}$ to find cos θ.

$$\begin{aligned}\cos\theta &= \frac{1}{\sec\theta}\\ &= \frac{1}{\frac{\sqrt{53}}{7}}\\ &= \frac{7}{\sqrt{53}}\\ &= \frac{7\sqrt{53}}{53}\end{aligned}$$

Verify each identity.

23. $\frac{\sin \theta}{1 - \cos \theta} + \frac{\sin \theta}{1 + \cos \theta} = 2 \csc \theta$

SOLUTION:

$\frac{\sin\theta}{1-\cos\theta} + \frac{\sin\theta}{1+\cos\theta}$

$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} + \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$ Common denominator

$= \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta} + \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$ Pythagorean Identity

$= \frac{1+\cos\theta}{\sin\theta} + \frac{1-\cos\theta}{\sin\theta}$ Divide out common factor of $\sin\theta$

$= \frac{2}{\sin\theta}$ Add.

$= 2\csc\theta$ Reciprocal Identity

24. $\frac{\cos \theta}{\sec \theta} + \frac{\sin \theta}{\csc \theta} = 1$

SOLUTION:

$\frac{\cos\theta}{\sec\theta} + \frac{\sin\theta}{\csc\theta}$

$= \cos\theta\cos\theta + \sin\theta\sin\theta$ Reciprocal Identities

$= \cos^2\theta + \sin^2\theta$ Simplify.

$= 1$ Pythagorean Identity

25. $\frac{\cot\theta}{1+\csc\theta}+\frac{1+\csc\theta}{\cot\theta}=2\sec\theta$

SOLUTION:

$$\frac{\cot\theta}{1+\csc\theta}+\frac{1+\csc\theta}{\cot\theta}$$

$$=\frac{\cot^2\theta}{\cot\theta(1+\csc\theta)}+\frac{(1+\csc\theta)^2}{\cot\theta(1+\csc\theta)}$$ Common denominator

$$=\frac{\cot^2\theta+(1+\csc\theta)^2}{\cot\theta(1+\csc\theta)}$$ Add.

$$=\frac{\csc^2\theta-1+1+2\csc\theta+\csc^2\theta}{\cot\theta(1+\csc\theta)}$$ Expand and use Pythagorean Identity.

$$=\frac{2\csc^2\theta+2\csc\theta}{\cot\theta(1+\csc\theta)}$$ Simplify.

$$=\frac{2\csc\theta(\csc\theta+1)}{\cot\theta(1+\csc\theta)}$$ Factor.

$$=\frac{2\csc\theta}{\cot\theta}$$ Divide out common factor.

$$=\frac{2}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}$$ Reciprocal/Quotient Identities

$$=\frac{2}{\cos\theta}$$ Multiply.

$$=2\sec\theta$$ Reciprocal Identity

26. $\frac{\cos\theta}{1-\sin\theta}=\frac{1+\sin\theta}{\cos\theta}$

SOLUTION:

$$\frac{\cos\theta}{1-\sin\theta}$$

$$=\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$ Multiply by conjugate, $1+\sin\theta$.

$$=\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$$ Pythagorean Identity.

$$=\frac{1+\sin\theta}{\cos\theta}$$ Divide out the common factor $\cos\theta$.

27. $\frac{\cot^2\theta}{1+\csc\theta} = \csc\theta - 1$

SOLUTION:

$$\begin{aligned}
&\frac{\cot^2\theta}{1+\csc\theta} \\
&= \frac{\cot^2\theta(1-\csc\theta)}{(1+\csc\theta)(1-\csc\theta)} && \text{Multiply by conjugate, } 1-\csc\theta. \\
&= \frac{\cot^2\theta(1-\csc\theta)}{1-\csc^2\theta} && \text{Simplify.} \\
&= \frac{\cot^2\theta(1-\csc\theta)}{-\cot^2\theta} && \text{Pythagorean Identity} \\
&= -(1-\csc\theta) && \text{Divide out the common factor of } \cot^2\theta. \\
&= \csc\theta - 1 && \text{Simplify.}
\end{aligned}$$

28. $\frac{\sec\theta}{\tan\theta} + \frac{\csc\theta}{\cot\theta} = \sec\theta + \csc\theta$

SOLUTION:

$$\begin{aligned}
&\frac{\sec\theta}{\tan\theta} + \frac{\csc\theta}{\cot\theta} \\
&= \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} + \frac{\frac{1}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}} && \text{Reciprocal/Quotient Identities} \\
&= \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta} && \text{Multiply by reciprocals.} \\
&= \frac{1}{\sin\theta} + \frac{1}{\cos\theta} && \text{Multiply.} \\
&= \csc\theta + \sec\theta && \text{Reciprocal Identities}
\end{aligned}$$

Find all solutions of each equation on the interval [0, 2π].

33. $2\sin x = \sqrt{2}$

SOLUTION:

$$\begin{aligned}
2\sin x &= \sqrt{2} \\
\sin x &= \frac{\sqrt{2}}{2}
\end{aligned}$$

On the interval [0, 2π), $\sin x = \frac{\sqrt{2}}{2}$ when $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

34. $4\cos^2 x = 3$

SOLUTION:

$$4\cos^2 x = 3$$
$$\cos^2 x = \frac{3}{4}$$
$$\cos x = \pm\frac{\sqrt{3}}{2}$$

On the interval [0, 2 π), $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}$ and $x = \frac{11\pi}{6}$ and $\cos x = -\frac{\sqrt{3}}{2}$ when $x = \frac{5\pi}{6}$ and $x = \frac{7\pi}{6}$.

35. $\tan^2 x - 3 = 0$

SOLUTION:

$\tan^2 x - 3 = 0$

$$\tan^2 x - 3 = 0$$
$$\tan^2 x = 3$$
$$\tan x = \pm\sqrt{3}$$

On the interval [0, 2π), $\tan x = \sqrt{3}$ when $x = \frac{\pi}{3}$ and $x = \frac{4\pi}{3}$ and $\tan x = -\sqrt{3}$ when $x = \frac{2\pi}{3}$ and $x = \frac{5\pi}{3}$.

36. $9 + \cot^2 x = 12$

SOLUTION:

$$9 + \cot^2 x = 12$$
$$\cot^2 x = 3$$
$$\cot x = \pm\sqrt{3}$$

On the interval [0, 2π), $\cot x = \sqrt{3}$ when $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$ and $\cot x = -\sqrt{3}$ when $x = \frac{5\pi}{6}$ and $x = \frac{11\pi}{6}$.

37. $2\sin^2 x = \sin x$

SOLUTION:

$$\begin{aligned} 2\sin^2 x &= \sin x \\ 2\sin^2 x - \sin x &= 0 \\ \sin x(2\sin x - 1) &= 0 \end{aligned}$$

$$\sin x = 0$$

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= \frac{1}{2} \end{aligned}$$

On the interval $[0, 2\pi)$, $\sin x = 0$ when $x = 0$ and π and $\sin x = \frac{1}{2}$ when $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

38. $3\cos x + 3 = \sin^2 x$

SOLUTION:

$$\begin{aligned} 3\cos x + 3 &= \sin^2 x \\ 3\cos x + 3 - \sin^2 x &= 0 \\ 3\cos x + 3 - (1 - \cos^2 x) &= 0 \\ \cos^2 x + 3\cos x + 2 &= 0 \\ (\cos x + 1)(\cos x + 2) &= 0 \end{aligned}$$

$$\begin{aligned} \cos x + 1 &= 0 \\ \cos x &= -1 \end{aligned}$$

$$\begin{aligned} \cos x + 2 &= 0 \\ \cos x &= -2 \end{aligned}$$

On the interval $[0, 2\pi)$, $\cos x = -1$ when $x = \pi$. The equation $\cos x = -2$ has no solution since the minimum value the cosine function can attain is -1.

Solve each equation for all values of *x*.

39. $\sin^2 x - \sin x = 0$

SOLUTION:

$$\sin^2 x - \sin x = 0$$
$$\sin x(\sin x - 1) = 0$$

$$\sin x = 0$$

$$\sin x - 1 = 0$$
$$\sin x = 1$$

The period of sine is 2π, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions to $\sin x = 0$ on this interval are 0 and π and the solution to $\sin x = 1$ on this interval is $\frac{\pi}{2}$.

Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of 2π. The solutions $x = 0 + 2n\pi$ and $x = \pi + 2n\pi$ can be combined to $x = n\pi$. Therefore, the general form of the solutions is $n\pi$, $\frac{\pi}{2} + 2n\pi$, where n is an integer.

40. $\tan^2 x = \tan x$

SOLUTION:

$$\tan^2 x = \tan x$$
$$\tan^2 x - \tan x = 0$$
$$\tan x(\tan x - 1) = 0$$

$$\tan x = 0$$

$$\tan x - 1 = 0$$
$$\tan x = 1$$

The period of tangent is π, so you only need to find solutions on the interval $[0, \pi)$. The solution to $\tan x = 0$ on this interval is 0 and the solution to $\tan x = 1$ on this interval is $\frac{\pi}{4}$.

Solutions on the interval $(-\infty, \infty)$ are found by adding integer multiples of π. Therefore, the general form of the solutions is $n\pi$, $\frac{\pi}{4} + n\pi$, where n is an integer.

43. $\sin^2 x = 1 - \cos x$

SOLUTION:

$$\sin^2 x = 1 - \cos x$$
$$1 - \cos^2 x = 1 - \cos x$$
$$\cos^2 x - \cos x = 0$$
$$\cos x(\cos x - 1) = 0$$

$$\cos x = 0$$

$$\cos x - 1 = 0$$
$$\cos x = 1$$

The period of cosine is 2π, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions to $\cos x = 0$ on this interval are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and the solution to $\cos x = 1$ on this interval is 0.

Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of 2π.

The solutions $x = \frac{\pi}{2} + 2n\pi$ and $x = \frac{3\pi}{2} + 2n\pi$ can be combined to $x = \frac{\pi}{2} + n\pi$.

Therefore, the general form of the solutions is $2n\pi$, $\frac{\pi}{2} + n\pi$, where n is an integer.

44. $\sin x = \cos x + 1$

SOLUTION:

$$\begin{aligned}\sin x &= \cos x + 1\\(\sin x)^2 &= (\cos x + 1)^2\\\sin^2 x &= \cos^2 x + 2\cos x + 1\\1 - \cos^2 x &= \cos^2 x + 2\cos x + 1\end{aligned}$$

$$2\cos^2 x + 2\cos x = 0$$
$$2\cos x(\cos x + 1) = 0$$

$$2\cos x = 0$$
$$\cos x = 0$$

$$\cos x + 1 = 0$$
$$\cos x = -1$$

The period of cosine is 2π, so you only need to find solutions on the interval $[0, 2\pi)$. The solutions to $\cos x = 0$ on this interval are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ and the solution to $\cos x = -1$ on this interval is π. Since each side of the equation was squared, check for extraneous solutions.

$$\sin x = \cos x + 1$$
$$\sin\left(\frac{\pi}{2}\right) \stackrel{?}{=} \cos\left(\frac{\pi}{2}\right) + 1$$
$$1 = 1$$
$$\sin x = \cos x + 1$$
$$\sin\left(\frac{3\pi}{2}\right) \stackrel{?}{=} \cos\left(\frac{3\pi}{2}\right) + 1$$
$$-1 \neq 1$$
$$\sin x = \cos x + 1$$
$$\sin \pi \stackrel{?}{=} \cos \pi + 1$$
$$0 = 0$$

Solutions on the interval $(-\infty, \infty)$, are found by adding integer multiples of 2π. Therefore, the general form of the solutions is $\frac{\pi}{2} + 2n\pi$, $\pi + 2n\pi$, where n is an integer.