Study Guide and Review - Chapter 2

Describe the end behavior of the graph of each polynomial function using limits. Explain your reasoning using the leading term test.

21. $f(x) = -3x^5 + 7x^4 + 3x^3 - 11x - 5$

SOLUTION:

$\lim_{x \to \infty} f(x) = -\infty$; $\lim_{x \to -\infty} f(x) = \infty$; The degree is odd and the leading coefficient is negative.

ANSWER:

$\lim_{x \to \infty} f(x) = -\infty$; $\lim_{x \to -\infty} f(x) = \infty$; The degree is odd and the leading coefficient is negative.

State the number of possible real zeros and turning points of each function. Then find all of the real zeros by factoring.

24. $F(x) = x^3 - 7x^2 + 12x$

SOLUTION:

The degree of $f(x)$ is 3, so it will have at most three real zeros and two turning points.

$$0 = x^3 - 7x^2 + 12x$$

$$0 = x(x^2 - 7x + 12)$$

$$0 = x(x - 4)(x - 3)$$

So, the zeros are 0, 3, and 4.

ANSWER:

3 real zeros and 2 turning points; 0, 3, and 4

25. $f(x) = x^5 + 8x^4 - 20x^3$

SOLUTION:

The degree of $f(x)$ is 5, so it will have at most five real zeros and four turning points.

$$0 = x^5 + 8x^4 - 20x^3$$

$$0 = x^3(x^2 + 8x - 20)$$

$$0 = x^3(x + 10)(x - 2)$$

So, the zeros are −10, 0, and 2.

ANSWER:

5 real zeros and 4 turning points; –10, 0, and 2

26. $f(x) = x^4 - 10x^2 + 9$

SOLUTION:

The degree of $f(x)$ is 4, so it will have at most four real zeros and three turning points.

Let $u = x^2$.

$$0 = \left(x^2\right)^2 - 10(x^2) + 9$$

$$0 = u^2 - 10u + 9$$

$$0 = (u - 9)(u - 1)$$

$$0 = (x^2 - 9)(x^2 - 1)$$

$$0 = (x + 3)(x - 3)(x + 1)(x - 1)$$

So, the zeros are –3, –1, 1, and 3.

ANSWER:

4 real zeros and 3 turning points; –3, –1, 1, and 3

For each function, (a) apply the leading term test, (b) find the zeros and state the multiplicity of any repeated zeros, (c) find a few additional points, and then (d) graph the function.

28. $f(x) = x^3(x - 3)(x + 4)^2$

SOLUTION:

a. The degree is 6, and the leading coefficient is 1. Because the degree is even and the leading coefficient is positive, $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$.

b. The zeros are 0, 3, and −4. The zero −4 has multiplicity 2 since $(x + 4)$ is a factor of the polynomial twice and the zero 0 has a multiplicity 3 since x is a factor of the polynomial three times.

c. Sample answer: Evaluate the function for a few x-values in its domain.

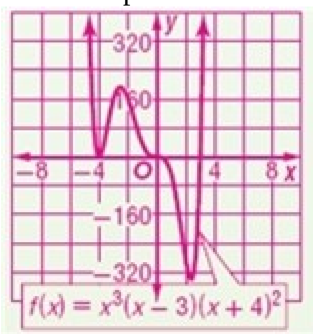
x	−1	0	1	2
$f(x)$	36	0	−50	−288

d.

Evaluate the function for several x-values in its domain.

x	−5	−4	−3	−2	3	4	5	6
$f(x)$	1000	0	162	160	0	4096	20,250	64,800

Use these points to construct a graph.



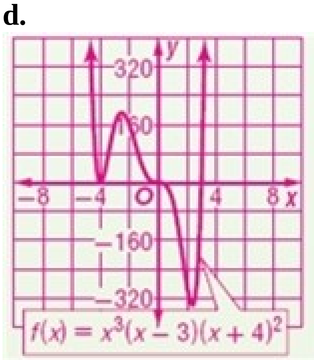
ANSWER:

a. $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to -\infty} f(x) = \infty$

b. 0 (multiplicity 3), 3, –4 (multiplicity 2)

c. Sample answer: (−1, 36), (0, 0), (1, −50), (2, −288)

d.



29. $f(x) = (x - 5)^2(x - 1)^2$

SOLUTION:

a. The degree is 4, and the leading coefficient is 1. Because the degree is even and the leading coefficient is positive, $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = \infty$.

b. The zeros are 5 and 1. Both zeros have multiplicity 2 because $(x - 5)$ and $(x - 1)$ are factors of the polynomial twice.

c. Sample answer: Evaluate the function for a few x-values in its domain.

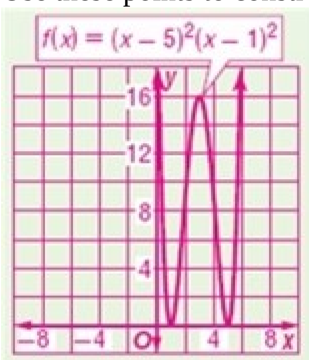
X	2	3	4	5
***f* (*x*)**	9	16	9	0

d.

Evaluate the function for several x-values in its domain.

X	−3	−2	−1	0	1	6	7	8
***f* (*x*)**	1024	441	144	25	0	25	144	441

Use these points to construct a graph.



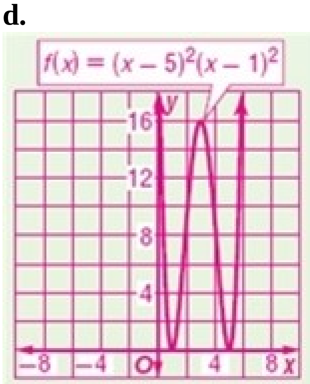
ANSWER:

a. $\lim_{x \to \infty} f(x) = \infty$; $\lim_{x \to -\infty} f(x) = \infty$

b. 5 (multiplicity 2), 1(multiplicity 2)

c. Sample answer: (2, 9), (3, 16), (4, 9), (5, 0)

d.



Divide using long division.

30. $(x^3 + 8x^2 - 5) \div (x - 2)$

SOLUTION:

$$
\begin{array}{r}
x^2 + 10x + 20 \\
x - 2\overline{)\,x^3 + 8x^2 + 0x - 5} \\
(-)\ \underline{x^3 - 2x^2} \qquad\qquad \\
10x^2 + 0x \qquad \\
(-)\ \underline{10x^2 - 20x} \qquad \\
20x - 5 \\
(-)\ \underline{20x - 40} \\
35
\end{array}
$$

So, $(x^3 + 8x^2 - 5) \div (x - 2) = x^2 + 10x + 20 + \frac{35}{x2}$.

ANSWER:

$x^2 + 10x + 20 + \frac{35}{x2}$

32. $(2x^5 + 5x^4 - 5x^3 + x^2 - 18x + 10) \div (2x - 1)$

SOLUTION:

$$
\begin{array}{r}
x^4 + 3x^3 - x^2 - 9 \\
2x - 1\overline{)\,2x^5 + 5x^4 - 5x^3 + x^2 - 18x + 10} \\
(-)\ \underline{2x^5 - x^4} \qquad\qquad\qquad\qquad\qquad \\
6x^4 - 5x^3 \qquad\qquad\qquad\qquad \\
(-)\ \underline{6x^4 - 3x^3} \qquad\qquad\qquad\qquad \\
-2x^3 + x^2 \qquad\qquad\qquad \\
(-)\ \underline{-2x^3 + x^2} \qquad\qquad\qquad \\
0 - 18x + 10 \\
(-)\ \underline{-18x + 9} \\
1
\end{array}
$$

So, $(2x^5 + 5x^4 - 5x^3 + x^2 - 18x + 10) \div (2x - 1) = x^4 + 3x^3 - x^2 - 9 + \frac{1}{2x-1}$.

ANSWER:

$x^4 + 3x^3 - x^2 - 9 + \frac{1}{2x-1}$

Divide using synthetic division.

35. $(2x^4 + 3x^3 - 10x^2 + 16x - 6) \div (2x - 1)$

SOLUTION:

Rewrite the division expression so that the divisor is of the form $x - c$.

$$\frac{2x^4 + 3x^3 - 10x^2 + 16x - 6}{2x - 1} = \frac{(2x^4 + 3x^3 - 10x^2 + 16x - 6) \div 2}{(2x - 1) \div 2}$$

$$= \frac{x^4 + \frac{3}{2}x^3 - 5x^2 + 8x - 3}{x - \frac{1}{2}}$$

Because $x - \frac{1}{2}, c = \frac{1}{2}$. Set up the synthetic division as follows. Then follow the synthetic division procedure.

$$\begin{array}{r|rrrrr} \frac{1}{2} & 1 & \frac{3}{2} & -5 & 8 & -3 \\ & & \frac{1}{2} & 1 & -2 & 3 \\ \hline & 1 & 2 & -4 & 6 & |0 \end{array}$$

The quotient is $x^3 + 2x^2 - 4x + 6$.

ANSWER:

$x^3 + 2x^2 - 4x + 6$

Use the Factor Theorem to determine if the binomials given are factors of $f(x)$. Use the binomials that are factors to write a factored form of $f(x)$.

38. $f(x) = x^4 - 2x^3 - 3x^2 + 4x + 4$; $(x + 1)$, $(x - 2)$

SOLUTION:

Use synthetic division to test each factor, $(x + 1)$ and $(x - 2)$.

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & -3 & 4 & 4 \\ & & -1 & 3 & 0 & -4 \\ \hline & 1 & -3 & 0 & 4 & |0 \end{array}$$

Because the remainder when $f(x)$ is divided by $(x + 1)$ is 0, $(x + 1)$ is a factor. Test the second factor, $(x - 2)$, with the depressed polynomial $x^3 - 3x^2 + 4$.

$$\begin{array}{r|rrrr} 2 & 1 & -3 & 0 & 4 \\ & & 2 & -2 & -4 \\ \hline & 1 & -1 & -2 & |0 \end{array}$$

Because the remainder when the depressed polynomial is divided by $(x - 2)$ is 0, $(x - 2)$ is a factor of $f(x)$. Because $(x + 1)$ and $(x - 2)$ are factors of $f(x)$, we can use the final quotient to write a factored form of $f(x)$ as $f(x) = (x + 1)(x - 2)(x^2 - x - 2)$. Factoring the quadratic expression yields $f(x) = (x - 2)^2(x + 1)^2$.

ANSWER:

yes; yes; $f(x) = (x - 2)^2(x + 1)^2$

40. $f(x) = x^4 + 5x^2 + 4$

SOLUTION:

Because the leading coefficient is 1, the possible rational zeros are the integer factors of the constant term 4. Therefore, the possible rational zeros of f are ±1, ±2, and ±4.
Using synthetic division, it does not appear that the polynomial has any rational zeros.
Factor $x^4 + 5x^2 + 4$. Let $u = x^2$.

$$0 = \left(x^2\right)^2 + 5(x^2) + 4$$

$$0 = u^2 + 5u + 4$$

$$0 = (u + 4)(u + 1)$$

$$0 = (x^2 + 4)(x^2 + 1)$$

Since $(x^2 + 4)$ and $(x^2 + 1)$ yield no real zeros, f has no rational zeros.

ANSWER:

±1, ±2, ±4; no rational zeros

Solve each equation.

42. $X^4 - 9x^3 + 29x^2 - 39x + 18 = 0$

SOLUTION:

Apply the Rational Zeros Theorem to find possible rational zeros of the equation. Because the leading coefficient is 1, the possible rational zeros are the integer factors of the constant term 18. Therefore, the possible rational zeros are ±1, ±2, ±3, ±6, ±9 and ±18.
By using synthetic division, it can be determined that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrrr} 1 & 1 & -9 & 29 & -39 & 18 \\ & & 1 & -8 & 21 & -18 \\ \hline & 1 & -8 & 21 & -18 & |0 \end{array}$$

By using synthetic division on the depressed polynomial, it can be determined that $x = 2$ is a rational zero.

$$\begin{array}{r|rrrr} 2 & 1 & -8 & 21 & -18 \\ & & 2 & -12 & 18 \\ \hline & 1 & -6 & 9 & |0 \end{array}$$

Because $(x - 1)$ and $(x - 2)$ are factors of the equation, we can use the final quotient to write a factored form as $0 = (x - 1)(x - 2)(x^2 - 6x + 9)$. Factoring the quadratic expression yields $0 = (x - 1)(x - 2)(x - 3)^2$. Thus, the solutions are 1, 2, and 3 (multiplicity: 2).

ANSWER:

1, 2, 3 (multiplicity: 2)

Find the domain of each function and the equations of the vertical or horizontal asymptotes, if any.

50. $f(x) = \dfrac{x(x3)}{(x5)2(x+3)2}$

SOLUTION:

The function is undefined at the real zeros of the denominator $b(x) = (x - 5)^2(x + 3)^2$. The real zeros of $b(x)$ are 5 and −3. Therefore, D = $\{x \mid x \neq 5, -3, x \in \mathbb{R}\}$.
Check for vertical asymptotes.
Determine whether $x = 5$ is a point of infinite discontinuity. Find the limit as x approaches 5 from the left and the right.

X	4.9	4.99	4.999	5	5.001	5.01	5.1
f (x)	15	1555	156,180	undef	156,320	1570	16

Because $\lim_{x \to 5^-} f(x) = \infty$ and $\lim_{x \to 5^+} f(x) = \infty$, $x = 5$ is a vertical asymptote of f.

Determine whether $x = -3$ is a point of infinite discontinuity. Find the limit as x approaches −3 from the left and the right.

X	−3.1	−3.01	−3.001	−3	−2.999	−2.99	−2.9
f (x)	29	2819	281,320	undef	281,180	2806	27

Because $\lim_{x \to -3^-} f(x) = \infty$ and $\lim_{x \to -3^+} f(x) = \infty$, $x = -3$ is a vertical asymptote of f.

Check for horizontal asymptotes. Use a table to examine the end behavior of $f(x)$.

x	−100	−50	−10	0	10	50	100
f (x)	0.0001	0.0004	0.0026	0	0.0166	0.0004	0.0001

The table suggests $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$. Therefore, you know that $y = 0$ is a horizontal asymptote of f.

ANSWER:

D = $\{x \mid x \neq 5, -3, x \in \mathbb{R}\}$; $x = 5$, $x = -3$, $y = 0$